correspondence with the r-card hands *allowing* repetition from the original k-card deck. The reader can supply the correspondence.

C. Communication Systems Development: Minimizing Range Code Acquisition Time,

R. C. Tausworthe

1. Introduction

The application of composite coding to ranging schemes (Refs. 1, 5 and 6) has produced systems capable of very accurate, relatively quick distance measurements, with large unambiguous range intervals. Optimization of these codes (Ref. 2) without constraint leads to a set of requirements on the component sequences not physically consistent, but does indicate what the important features are. For example, an unconstrained optimum would require all the components to be short PN sequences of the same period; but short codes combine to make longer ones only when their periods are different, and pseudonoise sequences only exist for periods of the form $p = 4N+3$.

Still, the optimization shows that the components should be short, relatively prime in length, and have distinguishable correlation functions.

Based on the supposition that all periods are equal, optimum choice of the encoding logic leads to a code having equal cross-correlation characteristics with each component sequence. Since the periods cannot be actually the same, there may sometimes be a better encoding logic than the one producing equal component correlations. This article investigates coding to minimize acquisition time under the constraint that all periods be different.

2. Probability of Correct Acquisition

The probability that a particular component having two-level autocorrelation with period is correctly acquired by maximum-likelihood techniques is given by

$$P = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-x^2} \left\{ \frac{1}{2} \left[1 + \operatorname{erf}(x+\beta) \right] \right\}^{p-1} dx$$

in which erf ( ) is the error function (Ref. 7) and β is defined by

$$\beta = C_{\max}(1 - C_{\min}/C_{\max}) \sqrt{\frac{ST}{N_0}}$$

The parameters appearing in β are

$\frac{S}{N_0}$ = The signal-power-to-noise-spectral-density ratio

T = The integration time-per-phase

$C_{\min}, C_{\max}$ = The minimum and maximum normalized cross-correlation values between the composite code and the component sequence

The total time required to correlate each phase sequentially is

$$T' = pT = \frac{(N_0/S)\, p\beta^2}{C^2_{\max}\,(1 - C_{\min}/C_{\max})^2}$$

which for an n-component code, yields a total acquisition time of

$$T_{\mathrm{acq}} = \sum_{i=1}^{n} T'_i = (N_0/S) \sum_{i=1}^{n} \frac{p_i \beta_i^2}{C^2_{\max(i)}\,[1 - C_{\min(i)}/C_{\max(i)}]^2}$$

The probability of correct acquisition is

$$P_{\mathrm{acq}} = \prod_{i=1}^{n} P_i$$

Under the assumption that it is desirable to acquire each component with equal likelihood, we set $P_i = P_j$ and have

$$P_i = P_{\mathrm{acq}}^{1/n}$$

3. Minimization of the Acquisition Time

Suppose we are given a desired P_{acq}; to achieve this, we compute each P_i, and translate this into a value of β_i. Suppose also that we elect to use PN sequences, for these maximize $(1 - C_{\min}/C_{\max})$ and thus tend to minimize T_{acq}. Under this latter assumption,

$$1 - C_{\min}/C_{\max} = (1+p)/p$$

Since T_{acq} decreases monotonically with each $C_{\max}$, we must find a set of $C_{\max(i)}$, subject to realizability, that are mutually as large as possible. It is known (Ref. 2) that the $C_{\max(i)}$ are constrained by the relation

$$\sum_{i=1}^{n} C_{\max(i)} \approx 2^{-n} \sum_{\mathbf{x}} f(\mathbf{x}) \left[n - 2 \,||\, \mathbf{x} \,||\right]$$

where $f(\mathbf{x})$ is the encoding logic function, $f(\mathbf{x}) = \pm 1$, $\mathbf{x}$ is a binary (0,1) vector truth-table variable, and $||\mathbf{x}||$ denotes the number of *ones* in $\mathbf{x}$. (The approximation is due only to an imbalance of $\pm$ 1's in the component sequences.) Note for example that $\sum C_{max}$ is greatest when $f(\mathbf{x})$ is a majority-vote logic, i.e. when $f(\mathbf{x}) = 1$ if $||\mathbf{x}|| < n/2$ and $f(\mathbf{x}) = -1$ if $||\mathbf{x}|| > n/2$. Tie votes when n is even are immaterial to the sum, but change individual values of $C_{max(i)}$.

We shall fix the sum $\sum C_{max(i)} = \alpha$ to find how each $C_{max(i)}$ should be related to the period p_i. Using standard Lagrange multiplier techniques, we define $\Phi = T_{acq} + \lambda\,(\sum C_{max(i)} - \alpha)$. Then optimum $C_{max(i)}$ values satisfy

$$\frac{\partial \Phi}{\partial C_{max(i)}} = -2\left(\frac{N_0}{S}\right)\left[\frac{p_i^3}{(1+p)^2}\right]\frac{\beta_i^2}{C^3_{max(i)}} + \lambda = 0$$

Hence we see that each $C_{max(i)}$ is related to the period p_i of its component sequence by

$$C_{max(i)} = K\,\beta_i^{2/3}\,\frac{p_i}{(1+p_i)^{2/3}}$$

For some constant K, evaluated to be

$$K = \alpha\left[\sum_i \beta_i^{2/3}\,\frac{p_i}{(1+p_i)^{2/3}}\right]^{-1}$$

The optimum value of T_{acq} can now be computed:

$$T_{acq}^{(opt)} = \frac{1}{\alpha^2}\left(\frac{N_0}{S}\right)\left\{\sum_{i=1}^{n}\left[\beta_i^{2/3}\,p_i/(1+p_i)^{2/3}\right]\right\}^3$$

$$= \alpha(N_0/S)/K^3$$

Example: Assume we are given the following:

$n = 6$ $\qquad p_3 = 11$

$N_0/S = -9.6\text{db} = .11$ $\qquad p_4 = 15$

$P_i = 0.99, i=1,\cdots,6$ $\qquad p_5 = 19$

$p_1 = 2$ $\qquad p_6 = 23$

$p_2 = 7$

The β_i^2 compute to be

$\beta_1^2 = 5.41$ $\qquad \beta_4^2 = 9.73$

$\beta_2^2 = 8.35$ $\qquad \beta_5^2 = 10.14$

$\beta_3^2 = 9.18$ $\qquad \beta_6^2 = 10.47$

In the usual case, and we shall assume it here, the first component having length $p_1 = 2$ is acquired differently than the rest; we shall further suppose that for practical reasons it is necessary to maintain 25% correlation with this component. With α set equal to its maximum value of 1.875, we can modify the previous theory to include optimization over the remaining five correlation values.

First we compute

$$K = \frac{1.625}{\sum_{i=2}^{6}\beta_i^{2/3}\,p_i/(1+p_i)^{2/3}} = \frac{1.625}{24.62} = 0.066$$

and then the desired values of correlation:

$C_{max(1)} = 0.250$ $\qquad C_{max(4)} = 0.333$

$C_{max(2)} = 0.234$ $\qquad C_{max(5)} = 0.368$

$C_{max(3)} = 0.290$ $\qquad C_{max(6)} = 0.399$

These are the values we would like to have if the encoding logic were capable of giving a continuous range of values. Being discrete, however, the set of correlation values most nearly equal to those above is obtained from the encoding logic whose truth-table appears in Table 1, which has

$C_{max(1)} = 0.2616$ $\qquad \Delta_1 = 0.0116$

$C_{max(2)} = 0.2498$ $\qquad \Delta_2 = 0.0158$

$C_{max(3)} = 0.3104$ $\qquad \Delta_3 = 0.0204$

$C_{max(4)} = 0.3143$ $\qquad \Delta_4 = -0.0187$

$C_{max(5)} = 0.3694$ $\qquad \Delta_5 = 0.0014$

$C_{max(6)} = 0.3918$ $\qquad \Delta_6 = -0.0072$

—an outstanding fit. In fact, the C_{max} values above are better than the desired ones listed previously; this is due to the fact that the constraint $\sum C_{max} = \alpha$ is only approximate, being affected by the imbalance of $\pm$ 1's in the

Table 1. An optimum ranging encoding logic

x_1	x_2	x_3	x_4	x_5	x_6	$f(x)$	x_1	x_2	x_3	x_4	x_5	x_6	$f(x)$
0	0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	1	0	1	0	0	0	0	1	0
0	0	0	0	1	0	0	1	0	0	0	1	0	0
0	0	0	0	1	1	0	1	0	0	0	1	1	0
0	0	0	1	0	0	0	1	0	0	1	0	0	0
0	0	0	1	0	1	0	1	0	0	1	0	1	0
0	0	0	1	1	0	0	1	0	0	1	1	0	1
0	0	0	1	1	1	1	1	0	0	1	1	1	1
0	0	1	0	0	0	0	1	0	1	0	0	0	0
0	0	1	0	0	1	0	1	0	1	0	0	1	0
0	0	1	0	1	0	0	1	0	1	0	1	0	1
0	0	1	0	1	1	1	1	0	1	0	1	1	1
0	0	1	1	0	0	0	1	0	1	1	0	0	0
0	0	1	1	0	1	1	1	0	1	1	0	1	1
0	0	1	1	1	0	1	1	0	1	1	1	0	1
0	0	1	1	1	1	1	1	0	1	1	1	1	1
0	1	0	0	0	0	0	1	1	0	0	0	0	0
0	1	0	0	0	1	0	1	1	0	0	0	1	1
0	1	0	0	1	0	0	1	1	0	0	1	0	0
0	1	0	0	1	1	1	1	1	0	0	1	1	1
0	1	0	1	0	0	0	1	1	0	1	0	0	0
0	1	0	1	0	1	1	1	1	0	1	0	1	1
0	1	0	1	1	0	0	1	1	0	1	1	0	1
0	1	0	1	1	1	1	1	1	0	1	1	1	1
0	1	1	0	0	0	0	1	1	1	0	0	0	1
0	1	1	0	0	1	0	1	1	1	0	0	1	1
0	1	1	0	1	0	0	1	1	1	0	1	0	1
0	1	1	0	1	1	1	1	1	1	0	1	1	1
0	1	1	1	0	0	0	1	1	1	1	0	0	1
0	1	1	1	0	1	1	1	1	1	1	0	1	1
0	1	1	1	1	0	1	1	1	1	1	1	0	1
0	1	1	1	1	1	1	1	1	1	1	1	1	1

component sequences. The values listed were obtained with each PN sequence having an excess -1.

Acquisition time of the latter five components is about 617 sec; the "ideal" values would require 621 sec; and a symmetric majority logic would require 735 seconds! The modified majority logic, in this case, affords a saving of almost two minutes over a symmetric majority.

D. Communication Systems Development: Cycle Slipping in Phase-Locked Loops,

R. C. Tausworthe

1. Introduction

Cycle slipping in phase-locked receivers is an observed phenomenon that is unpredictable and not accounted for by linear theories of loop operation. Viterbi (Ref. 8) was able to solve a Fokker-Planck equation to arrive at the expected time from lock to a slipped cycle for a first-order loop; however, extensions to higher-order loops by these methods have not, as yet, been forthcoming.

This article shows that the expected first-slip time of a loop of arbitrary order satisfies a linear differential equation reducible to one of the first order, to which formal solutions, at least, are easily given. The derivation follows directly from a random-walk model not restricted to processes of the Markov type. Computation of an exact solution involves being able to evaluate a certain conditional expectation, which ordinarily requires a prior solution for the probability function of the phase-error process. For the first-order loop, however, the expectation can be evaluated directly without computing $p(\phi, \dot{\phi})$, so the method yields the exact result. Rather than attempting to compute $p(\phi, \dot{\phi})$, for higher-order loops (which is still an unsolved problem), this article presents an approximate evaluation of the expectation valid for loops of any order. Specific behavior of second-order loop cycle-slipping is evaluated.

2. The Random-Walk Model

Let $\{x(t)\}$ be a stochastic process, and let t_0 be any observed time such that for a particular sample function $x(t)$ we have $|x(t_0)| < \lambda$, where λ is a given constant. Let $\tau(x_0)$ be the time required for the sample function starting from x_0 at t_0 to reach or pass the limits $\pm\lambda$ for the first time; that is, $\tau(x_0)$ is the random variable defined by

$$\tau(x_0) = \sup\{\tau; |x(t_0 + \delta)| < \lambda \text{ for all } 0 \leq \delta < \tau\} \quad (1)$$

Given t_0 and x_0, the mean value of $\tau(x_0)$ will be denoted

$$T(x_0) = E\{\tau(x_0) \mid t_0, x_0\} \quad (2)$$

Now let $\Delta t > 0$ be given, and define $\Delta x_0 = x(t_0 + \Delta t) - x(t_0)$; naturally Δx_0 is a random variable depending on